Philosophy & mathematics

Determinate Preservation

Hegelian Sublation and the Homotopy of Resolving Levels

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Abstract

Lawvere's categorical rendering of Hegelian sublation makes preservation precise through least resolving levels. We study two choices on which such leastness depends: the relations represented and the coefficient semantics governing generation and reconstruction. For diagrams over a finite poset, a matching-map argument gives the least full reconstruction subset for any specified diagram family, and hence for arbitrary starting levels. For constant generation this becomes an upper-interval criterion. We compute coefficient-relative resolution profiles from sets through truncated homotopy types. A Boolean family has explicit wedge-of-sphere obstructions and arbitrarily many distinct least levels, although every earlier resolution comparison remains unchanged. A compact family realizes m distinct levels on $2m + 2$ objects; for an initial one-object level, the first set-to-groupoid jump requires exactly six. The jump also survives non-constant generation. Without preservation of generation, a second family reverses the inclusion: richer coefficients require fewer positions. A further comparison, grounded in an interactive reader of the *Science of Logic*, orders the same positions by ancestry or succession. Both profiles are invariant under coefficient enrichment, yet their least subsets differ sharply. Together these calculations show why old-test agreement does not secure later leastness, and why coefficient stability does not establish an interpretation. Reading these results alongside Hegel's accounts of retention, mediation, and method, we defend determinate preservation: a reconstruction must make its asserted mode of retention and its interpretive warrant answerable together. The resulting comparison with nLab and contemporary formalizations separates mathematical necessity within a representation from justification of the relations and tests that produce it.

Keywords: Hegel; Science of Logic; Aufhebung; categorical logic; Kan extension; truncated spaces; philosophy of formalization.

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1 The question hidden inside preservation

A mathematical reconstruction of Hegelian sublation makes two commitments: it selects relations that represent the content, and it selects tests under which that content counts as preserved. Universal properties can determine a result once those commitments are fixed. They do not by themselves establish that the commitments express the intended mediation. We investigate this distinction through exact calculations of least resolving levels. The calculations show both how an expanded test domain can change a least result while preserving every earlier comparison, and how stability across test domains can leave the interpretation of the represented relations unsettled.

For the first comparison, the relevant enrichment replaces sets by homotopy types. A set distinguishes its elements; a homotopy type also allows identifications between elements, identifications between those identifications, and further coherent structure. The issue is therefore not merely whether the same symbols remain available in a more elaborate foundation. The universal constructions denoted by those symbols can change. A compatibility condition that is sufficient for sets can leave a detectable obstruction when paths and higher paths are admitted.

The philosophical importance of the question comes from sublation. In the *Science of Logic*, Hegel distinguishes the result of mediation from an empty disappearance. In the remark on the expression *Aufheben*, the earlier determination persists in a changed status: its immediacy is cancelled, but it is not simply annihilated [1]. Whatever else one takes this to mean, a mathematical reconstruction should explain which aspect of the earlier determination remains available and in what sense. The bare word "preservation" does not yet provide that explanation.

Lawvere's theory supplies a demanding candidate. A level has both a skeletal and a sheaf presentation in an ambient topos. A higher level resolves it when the earlier skeleta become sheaves at the higher level. An Aufhebung is a *least* resolving level. This is substantially stronger than enclosing two objects in a third. The construction has a mathematical order, an explicit factorization condition, and a minimality claim [2, 3]. The question we ask is whether such leastness is unchanged when the coefficients become homotopical.

Our answer is negative in a controlled and quantitative sense. We keep the finite indexing poset and the initial level fixed. We change only the category of coefficients, from sets to groupoids and then to successively higher truncated spaces. Every earlier resolution comparison remains unchanged, including the construction of the objects being tested. Nevertheless the least resolving full-subcategory level changes at every stage until the whole indexing category is required. Agreement on every earlier test does not therefore secure stability of the least adequate reconstruction. This is the stronger claim we defend, beyond a general reminder that models have parameters.

1.1 The result in advance

For $N \geq 3$, let $\mathcal{P}_N$ be the poset of nonempty proper subsets of $[N] = \{1, \ldots, N\}$. Place two objects $b < a$ below every element of $\mathcal{P}_N$, obtaining a poset $\mathcal{C}_N$ with 2^N objects. The starting level is $D = \{b\}$. We work with covariant functor categories, equivalently with presheaves on the opposite posets. Because b is initial, extension from D produces constant diagrams.

For $k \geq 0$, let $\mathcal{V}_k = \mathcal{S}_{\leq k}$ be the category of k-truncated spaces. Thus $\mathcal{V}_0 \simeq \mathbf{Set}$, while $\mathcal{V}_1$ may be presented by groupoids. A full subposet R containing b resolves D if every constant $\mathcal{V}_k$-valued diagram on $\mathcal{C}_N$ is a right Kan extension of its restriction to R. The main theorem

computes the least such R:

$$\mathsf{A}_k(D) = \begin{cases} \{b\} \cup \{S \in \mathcal{P}_N : |S| \geq N-k-2\}, & 0 \leq k \leq N-3, \\ \mathcal{C}_N, & k \geq N-2. \end{cases} \tag{1.1}$$

For unrestricted spaces the answer is also $\mathcal{C}_N$. The word "least" here refers to *all* full subposets containing b, not merely to a preselected chain of rank cuts. The restricted class of levels is explicit; we do not assume a classification of all higher essential subtoposes.

The proof identifies the comparison map at an object x with

$$A \longrightarrow \mathrm{Map}(\mathrm{B}(x \downarrow R), A).$$

For a proposed resolver, the omitted information is therefore encoded by a comma-category nerve. At a, the nerve for the first line of (1.1) is a wedge of $\binom{N-1}{k+2}$ spheres of dimension $k+1$. These spheres are invisible to k-types but visible to $(k+1)$-types. The theorem is consequently more informative than an isolated failure of transfer: it specifies how much coherence each resolution leaves untested.

1.2 What is and is not claimed as a contribution

The topology of simplex skeleta, the formula for a limit of a constant space-valued diagram, and the relation between truncation and connectivity are standard. They are proved or recalled below and are not proposed as new foundational results. The contribution offered here is a worked theory of finite coefficient-relative resolution profiles: a matching-support formula, explicit leastness calculations, a linear-size family with a sharp six-object first jump, a nonconstant extension, and a reverse family in which changed generation makes the least resolver smaller. The formula is an elementary consequence of Reedy reconstruction, not a new foundation for it. The philosophical argument pairs the changed-test calculation with a reader-based comparison in which coefficient stability fails to settle the interpretation of relations. These comparisons give distinct, precise tasks to the requirement of determinate preservation.

The finite calculations have a methodological use. The paper does not solve the general problem of transporting Aufhebung between arbitrary topoi. It does not derive the sequence of Hegel's categories, and it does not identify the sphere in an example with a contradiction in Hegel's text. Its claim is that the family supplies a counterexample to a tempting methodological inference: mathematical leastness in one semantic setting does not, by itself, establish leastness in a richer one.

The relevant background is wider than the vocabulary of Aufhebung. Truncated cofinality already explains why higher coefficients require more coherent limit data [4]. We use that background rather than claiming to rediscover it. What remains to be assessed by further specialist comparison is whether the specific finite resolution family has appeared in another form. The argument for its philosophical significance does not depend on presenting its elementary ingredients as previously unknown.

The ordinary finite-poset endpoint also meets recent work on minimal initial functors. Dey and Lesnick describe a least full subposet sufficient for ordinary limit computation, as well as finer minimality questions allowing relations to be removed [5]. Our coefficient-indexed profile should be compared with that work and with truncated cofinality, not presented as a new general theory of ordinary minimal limit computation.

2 Hegel, Lawvere, and the burden of a formal reconstruction

2.1 Three requirements that should not be collapsed

The Introduction to the *Science of Logic* connects determinate negation with a result that has a specific content because of what it negates. Its advance is not the replacement of one empty designation by another. The method is also not supposed to be an independently supplied device applied to an indifferent subject matter [6]. These commitments motivate three distinct questions for a reconstruction.

First, there is a question of *retention*. What survives the transition? An embedding may preserve objects while changing the morphisms between them. A quotient may preserve connected components while erasing loops. A logical translation may preserve provable sentences while adding new inferential relations. None of these is preservation without qualification. A reconstruction must say which of these achievements, or which other achievement, is relevant to the passage under interpretation.

Second, there is a question of *determination*. Why is this result the result of this earlier content? A construction that can be attached equally well to any pair of terms may be mathematically legitimate but explanatorily weak. What matters is not merely that something later can accommodate something earlier. The earlier structure should contribute to specifying the later one.

Third, there is a question of *necessity*. What excludes alternatives? A universal property can answer this question after the ambient category, admissible constructions, and relevant equivalences have been fixed. Whether those choices express the philosophical content is an additional question. Confusing the two questions turns the uniqueness of an answer to a specified problem into an argument that the problem was specified in the only adequate way.

These requirements do not amount to a definition of Hegel's method. They are obligations incurred by a mathematical interpretation that claims to represent its preservation, determination, or necessity. The distinction is important. An interpretation can meet one obligation and fail another; it can also be mathematically productive while making only a modest claim about Hegel.

2.2 Why the categorical proposal deserves a serious test

Lawvere's use of adjoint structures gives the philosophical vocabulary mathematical consequences. His account of unity and identity of opposites studies common retractions and their adjoint forms in examples from calculus and physics [7]. The value of this approach lies in the possibility of asking whether the proposed relation exists, how it behaves, and which further structures it entails. The resulting mathematics can resist an interpretation as well as illustrate it.

For full subcategories of an indexing category, restriction has left and right Kan extension adjoints. These two extensions produce different presentations of the same smaller diagram category inside the larger one. A resolving level requires the presentation supplied on one side of the earlier adjunction to be accessible through the other side of the later one. Leastness then constrains how much additional structure is admitted. In this respect, Lawvere's formulation directly addresses determination: the earlier skeletal image constrains which levels can count as resolutions.

The formulation also makes a hidden choice visible. A Kan extension is computed in a category of coefficients. In sets its limits enforce ordinary compatibility. In spaces the same notation requests coherent compatibility. The latter includes data that the former cannot

detect. The notation can remain unchanged while the mathematical demand becomes stronger. Our calculation makes this dependence explicit.

2.3 A textual benchmark: from becoming to determinate being

The transition at the beginning of the *Logic* provides a demanding benchmark. In the account of coming-to-be and ceasing-to-be, each moment's direction passes into the other. Becoming depends on their distinction, yet that distinction is itself unstable; its result is not an empty return to nothing [8]. The subsequent remark makes retention more specific: the earlier determinations become moments whose significance is established within their new unity [1].

Two requirements follow for an interpretation of this particular passage. It must explain how the earlier content supplies a reason for its transformation, and how the retained determinations acquire their changed significance. Merely putting copies of two objects into a third would meet neither requirement. Exact recovery of an earlier object would meet a retention condition, but would still not explain its changed status or the need for the transition.

Our model leaves those two requirements open. Its indexing relations are supplied in advance, and admitting the next coefficient category is a decision in the construction. The theorem then forces a different least resolver. It establishes the conditional necessity of a repair, not the self-generated necessity of Hegel's transition. The test separates a recoverability achievement from the stronger interpretive claims that could otherwise be attached to it.

Hegel's Introduction itself warns against importing mathematical method into philosophy [6, online paragraph 62]. Our use of mathematics is diagnostic: it examines a proposed formal correlate. It does not claim that a calculation generates the movement of the *Logic*. This addresses the warning without pretending to settle Hegel's wider assessment of mathematics.

2.4 The nLab project and its stated restrictions

nLab's *Science of Logic* entry develops a modal and type-theoretic reconstruction: concepts are treated as types, moments through modalities, and opposition through adjoint structures. It connects an abstract treatment with universes and categorical models. It also places restrictions on its own results, distinguishing a 0-type argument about determinate negation from stable-type constructions [9]. The separate Aufhebung entry states a least resolving-level condition [10].

These restrictions are examples of the discipline advocated here, not errors that our finite example corrects. Likewise, nLab's claim that adjoints are determined is expressly conditional on a concrete interpreted moment. The question is how to extend comparable explicitness to the coefficient dependence of least resolving levels. Prähauser's exposition distinguishes a broader resolving relation from a unique least resolution [11]; terminology alone cannot supply minimality.

The present test is therefore internal to a mathematical possibility the project itself takes seriously: movement between categorical and homotopical settings. It is not an objection that a reconstruction fails to preserve every distinction in every conceivable foundation. Rather, when a reconstruction uses both ordinary and higher versions of its structures, their agreement must be established where it is invoked. A restricted theorem remains restricted even if its notation resembles a more general theorem.

There is a constructive way to respond. One can retain the ordinary interpretation and justify its restriction to sets. One can strengthen the higher reconstruction and supply the extra coherence it requires. Or one can treat the sequence of coefficient categories as a family of distinct interpretations with distinct explanatory purposes. Each response is more informative than assuming that a familiar universal property carries its philosophical force unchanged across the passage.

2.5 Formalization as a source of philosophical questions

One might ask whether all this is merely a technical qualification on an analogy. It would be, if the claim under examination were only that adjunctions resemble some aspects of opposition. The stakes change when the claim is that a canonical or necessary transition has been exhibited. At that point, the conditions of canonicity become part of the philosophical argument.

Hegel's discussion of determinate being offers a useful restraint here. He distinguishes determinations present in the concept under development from those supplied by the interpreter's comparison [12]. A homotopical reconstruction need not deny that distinction. It should apply it to itself. A path, a higher identity, or an ambient infinity-topos is initially a resource of the reconstruction. Its relevance to the interpreted content requires an argument. The mathematical theorem can sharpen that argument by showing exactly what depends on the resource, but cannot supply the historical or philosophical identification on its own.

3 Coefficient-relative resolution

All indexing categories in the main construction are finite ordinary posets, regarded as categories. Their nerves are also used as indexing infinity-categories when the coefficients are spaces. Write $\mathrm{B}J = |\mathrm{N}J|$ for the classifying space of an ordinary category J, with $\mathrm{B}\varnothing = \varnothing$.

3.1 Conventions and Kan extensions

For $k \geq 0$ put $\mathcal{V}_k = \mathcal{S}_{\leq k}$, and put $\mathcal{V}_\infty = \mathcal{S}$. These categories admit the limits and colimits used below. Limits of k-types agree with their limits in spaces; colimits are obtained by taking the colimit in spaces and then truncating. When groupoids represent 1-types, their limits here are homotopy-coherent limits, not strict limits in the ordinary category of groupoids. We use *covariant* diagrams $\mathrm{Fun}(C, \mathcal{V}_k)$. This is the presheaf convention on C^{op}, not on C.

For a full inclusion $u : R \hookrightarrow C$, let u^* be restriction. Its adjoints are

$$u_! = \mathrm{Lan}_u \;\dashv\; u^* \;\dashv\; u_* = \mathrm{Ran}_u\,. \tag{3.1}$$

Both extension functors are fully faithful. On the ambient diagram category define

$$\mathrm{sk}_R = u_! u^*, \qquad \mathrm{cosk}_R = u_* u^*.$$

The counit $\mathrm{sk}_R X \to X$ detects R-skeletal objects; the unit $X \to \mathrm{cosk}_R X$ detects R-coskeletal objects. If $D \subseteq R$, the skeletal and coskeletal presentations for D factor through those for R, by composition of Kan extensions. This is the ordering of the induced levels used throughout the paper.

Definition 3.1. Let $D \subseteq R \subseteq C$ be full subcategories. The level R *k-resolves* D if for every $A : D \to \mathcal{V}_k$ the canonical unit

$$\mathrm{Lan}_D A \longrightarrow \mathrm{Ran}_R \,\mathrm{res}_R \,\mathrm{Lan}_D A \tag{3.2}$$

is an equivalence. A *coefficient-relative Aufhebung*, written $\mathsf{A}_k(D)$, is a least such full subcategory under inclusion, if it exists. The definition for $k = \infty$ uses unrestricted spaces.

The qualifier records a substantive restriction. For $k = 0$, this is the usual skeletal-to-coskeletal resolution condition on the specified ordinary presheaf levels. For higher k, we extend the condition through (3.1). We do not silently identify the admissible family of induced levels with every possible essential localization of a higher category of diagrams.

3.2 Matching maps and least reconstruction subsets

The existence of a least full resolver over a finite poset can be separated from the calculation of that resolver. Let $\mathcal{V}$ be an infinity-category with finite limits. For $X : C \to \mathcal{V}$, write

$$M_x X = \lim_{y \in C_{>x}} X(y), \qquad S(X) = \{x \in C : X(x) \longrightarrow M_x X \text{ is not an equivalence}\}. \tag{3.3}$$

The map is induced by the arrows out of x; at a maximal element its target is terminal. These are matching objects for the inverse Reedy structure obtained by assigning smaller degree to higher objects of C. Matching objects and inductive extension of diagrams are standard Reedy machinery [13, section 3]. The following elementary consequence is useful because it applies to arbitrary full subposets, not just degree cuts.

Proposition 3.2 (Least reconstruction subset)**.** *Let C be a finite poset, $R \subseteq C$ full, and $X : C \to \mathcal{V}$. The canonical unit $X \to \mathrm{Ran}_R(X|_R)$ is an equivalence if and only if $S(X) \subseteq R$. Consequently, for any specified family $\mathcal{F}$ of such diagrams, the least full subposet from which every member is right Kan extended is*

$$S(\mathcal{F}) = \bigcup_{X \in \mathcal{F}} S(X). \tag{3.4}$$

Proof. If X is right Kan extended from R and $x \notin R$, factor the inclusion through $C \setminus \{x\}$. Composition of right Kan extensions shows that X is also right Kan extended from this intermediate subposet. Its unit at the only omitted object is exactly $X(x) \to M_x X$, because $(x \downarrow (C \setminus \{x\})) = C_{>x}$. Thus $x \notin S(X)$.

Conversely, suppose $S(X) \subseteq R$. List the objects of $C \setminus R$ in an increasing linear extension, and delete them in that order. When x is deleted, every $y > x$ is still present. The comparison for right extension from the remaining subposet is an equivalence at every retained object, and at x it is the original matching map $X(x) \to M_x X$. This is an equivalence by assumption. Compose the finitely many right Kan extensions to obtain the assertion for R. Applying the equivalence to every $X \in \mathcal{F}$ proves the formula for a family. □

Corollary 3.3 (Arbitrary starting levels)**.** *If $D \subseteq C$ is any full subposet and $\mathcal{V}$ also admits the left Kan extensions in question, the least full resolver of D exists and is*

$$D \cup \bigcup_{A \in \mathrm{Fun}(D, \mathcal{V})} S(\mathrm{Lan}_D A). \tag{3.5}$$

In particular this holds for every $\mathcal{V}_k$ used here.

This is an existence result, not a general algorithm: the quantifier over A can be substantial. It also locates the role of leastness in our finite setting. Once a diagram family and its matching tests are fixed, the least subset is already forced; the substantive problem is to compute it and to justify the chosen family. No new general theory of Reedy matching objects is claimed.

Example 3.4 (Why the finiteness hypothesis matters)**.** Take $C = \mathbb{N}$ with its usual order and $D = \{0\}$. Every strict upper interval has a least element, so the matching support of every constant diagram is empty. Nevertheless a full R containing 0 resolves constants exactly when it is unbounded: then each $(n \downarrow R)$ has a least element, while for bounded R an upper comma is eventually empty. There is no inclusion-minimal unbounded subset containing 0, because any positive member may be deleted. Thus the finite matching-support conclusion cannot be extended to arbitrary infinite posets merely by using the same formula.

3.3 Resolution of an initial one-object level

Let C have an initial object b, and let $D = \{b\}$. Since the comma category $(D \downarrow x)$ has one object and its identity for every x, left extension from D is the constant diagram:

$$\mathrm{Lan}_D A \simeq \Delta_C A. \tag{3.6}$$

This holds for every coefficient category under consideration, including arbitrary spaces. There is no homotopy-colimit defect in the left extension of this particular starting level.

Lemma 3.5 (Constant-diagram criterion)**.** *Let $R \subseteq C$ be full and contain b. For finite $k \geq 0$, the following are equivalent:*

(i) R k-resolves D;
(ii) $\tau_{\leq k}\mathrm{B}(x \downarrow R) \simeq *$ for every $x \in C$.

For $k = \infty$, replace (ii) by contractibility of every $\mathrm{B}(x \downarrow R)$.

Proof. Pointwise right Kan extension and (3.6) identify the comparison at x with

$$A \longrightarrow \lim_{(x \downarrow R)} \Delta A \simeq \mathrm{Map}(\mathrm{B}(x \downarrow R), A). \tag{3.7}$$

The equivalence is the standard formula for a limit of a constant diagram of spaces. The map sends a point of A to the corresponding constant map. For A a k-type, the target depends only on $\tau_{\leq k}\mathrm{B}(x \downarrow R)$. Thus (ii) implies (i). Conversely, if (3.7) is an equivalence for every $A \in \mathcal{V}_k$, the map $\tau_{\leq k}\mathrm{B}(x \downarrow R) \to *$ induces equivalences on mapping spaces into every object of $\mathcal{V}_k$. Yoneda implies that it is an equivalence. The same argument without truncation proves the unrestricted case. Equivalences of diagrams are detected pointwise. □

Our connectivity convention is that a nonempty space T is *k-connected* when $\tau_{\leq k}T \simeq *$. Thus 0-connected means connected, and 1-connected means simply connected. We state the convention because cofinality references can shift the index by one. The lemma is a constant-diagram instance of truncated cofinality [4]; the short proof fixes exactly the special case needed here.

For sets the criterion sees only connected components. For groupoids it also sees the fundamental group. For unrestricted spaces it requires the whole homotopy type of each comma nerve to be trivial. An ordinary resolution can therefore leave an obstruction detectable at the next coefficient level.

3.4 An intrinsic criterion for finite posets

For constant diagrams, the matching-map calculation specializes to a criterion depending only on the indexing poset. It is also a form of truncated cofinality [4]; the matching argument supplies a direct proof without an additional cofinality theorem.

For a finite poset C, let $C_{>x}$ be the strict upper interval at x. Define

$$E_k(C) = \{x \in C : \tau_{\leq k}\mathrm{B}(C_{>x}) \not\simeq *\}. \tag{3.8}$$

For $k = \infty$, replace the condition by noncontractibility. Every maximal element belongs to $E_k(C)$, because its upper interval is empty.

Theorem 3.6 (Upper-interval criterion)**.** *For finite $k \geq 0$, a finite poset C, and a full subposet R,*

$$\big[\tau_{\leq k}\mathrm{B}(x \downarrow R) \simeq * \text{ for every } x \in C\big] \quad \Longleftrightarrow \quad E_k(C) \subseteq R. \tag{3.9}$$

The analogous statement holds without truncation for $k = \infty$. If b is initial and $D = \{b\}$, it follows that

$$\mathsf{A}_k(D) = \{b\} \cup E_k(C) \tag{3.10}$$

among full-subcategory levels containing b.

Proof. For $X = \Delta_C A$, the matching map at x is $A \to \mathrm{Map}(\mathrm{B}(C_{>x}), A)$. As A ranges over all k-types, Yoneda identifies the union of their matching supports with $E_k(C)$. For unrestricted spaces the same argument uses no truncation. Apply proposition 3.2 to this family: right extension from R recovers every constant diagram exactly when $E_k(C) \subseteq R$. The pointwise mapping-space calculation in the proof of lemma 3.5, which does not require an initial object, identifies it with the comma-nerve condition in (3.9). If b is initial, left generation from D is constant, so corollary 3.3 gives the last formula. □

For $k = 0$, the dual lower-interval description is already part of the theory of least full initial subposets: see Dey–Lesnick [5, Remark 3.6] and Brüstle–Desrochers–Leblanc [14, Theorem C and Remark 2]. The higher criterion is the matching-map consequence just proved, consistent with standard truncated cofinality. The untruncated reasoning is related to established point-deletion methods for finite spaces [15]. We use the intrinsic form to organize and calculate a resolution profile, rather than claiming a new cofinality theorem.

4 A family with an exact resolution profile

4.1 The indexing posets

Fix $N \geq 3$. Let $\mathcal{P}_N$ consist of nonempty proper subsets of $[N]$, ordered by inclusion. Define

$$\mathcal{C}_N = \{b < a\} \star \mathcal{P}_N, \tag{4.1}$$

where this means $b < a < S$ for every $S \in \mathcal{P}_N$, with the original inclusion order on $\mathcal{P}_N$. Thus $|\mathcal{C}_N| = 2^N$. The object b is initial. We compare all full subposets containing $D = \{b\}$.

For $0 \leq k \leq N - 3$, define

$$R_k = \{b\} \cup \{S \in \mathcal{P}_N : |S| \geq N - k - 2\}. \tag{4.2}$$

The object a is omitted. The phrase "top $k + 2$ ranks" refers to cardinalities $N - k - 2, \ldots, N - 1$, not to the existence of a largest element. The proper Boolean poset has several maximal elements.

4.2 The standard topological ingredient

Lemma 4.1 (Simplex skeleton)**.** *Let Δ^{m-1} be the simplex on m vertices, and let $1 \le q \le m-2$. Its q-skeleton has homotopy type*

$$\bigvee^{\binom{m-1}{q+1}} S^q. \tag{4.3}$$

It is $(q-1)$-connected and has nonzero reduced integral homology in degree q. The order complex of the poset of nonempty proper subsets of an m-element set is S^{m-2} for $m \ge 2$.

Proof. Write $K = \mathrm{sk}_q \Delta^{m-1}$ and choose a vertex v. Its closed star is $L = v * \mathrm{sk}_{q-1} \Delta^{m-2}$, a contractible subcomplex. Every simplex of dimension less than q belongs to L. The q-simplices outside L are exactly those not containing v, of which there are $\binom{m-1}{q+1}$. Their boundaries belong to L. Collapsing L gives the stated wedge. Since $L \hookrightarrow K$ is a cofibration and L is nonempty and contractible, the quotient map is a homotopy equivalence. For the final assertion, the order complex is the barycentric subdivision of the boundary of an $(m-1)$-simplex. □

This is elementary standard topology. The proof makes the quantitative calculation independent of a black-box connectivity assertion.

Lemma 4.2 (The omitted-face nerves)**.** *Let $0 \le k \le N-3$. If $x \in R_k$, then $\mathrm{B}(x \downarrow R_k)$ is contractible. At $x = a$ it is homotopy equivalent to*

$$\bigvee^{\binom{N-1}{k+2}} S^{k+1}. \tag{4.4}$$

If $x = S \in \mathcal{P}_N \setminus R_k$ and $m = N - |S|$, then

$$\mathrm{B}(S \downarrow R_k) \simeq \bigvee^{\binom{m-1}{k+2}} S^{k+1}. \tag{4.5}$$

Proof. For $x \in R_k$, the identity arrow is initial in the comma category. For a, the comma category is the poset of proper subsets of $[N]$ of cardinality at least $N-k-2$. Complementation reverses its order and gives the poset of nonempty subsets of $[N]$ of cardinality at most $k+2$. Its order complex is the barycentric subdivision of $\mathrm{sk}_{k+1} \Delta^{N-1}$. Reversing a poset does not change the homeomorphism type of its order complex. Apply lemma 4.1.

For an omitted S, the comma category consists of selected supersets $T \supset S$. Taking complements in $[N]$ identifies its opposite with the nonempty subsets of $[N] \setminus S$ of cardinality at most $k+2$. Here $m \ge k+3$, since S is omitted. Apply the same argument to $\mathrm{sk}_{k+1} \Delta^{m-1}$. □

4.3 The main theorem

Theorem 4.3 (Exact coefficient-relative Aufhebung)**.** *For $D = \{b\}$ in $\mathcal{C}_N$, the least resolving full subposet exists at every coefficient level. It is*

$$\mathsf{A}_k(D) = \begin{cases} R_k, & 0 \le k \le N-3, \\ \mathcal{C}_N, & N-2 \le k \le \infty. \end{cases} \tag{4.6}$$

In particular the levels at $k = 0, 1, \ldots, N-2$ form a strictly increasing chain of $N-1$ different least resolutions.

Proof. Compute the set $E_k(\mathcal{C}_N)$ in theorem 3.6. The strict upper interval of b has initial object a, so b never belongs to this essential set. The strict upper interval of a is $\mathcal{P}_N$, with classifying space S^{N-2}. Hence a belongs exactly when $k \geq N-2$, or when $k = \infty$.

For $S \in \mathcal{P}_N$, let $m = N - |S|$. If $m = 1$, its strict upper interval is empty, so S is always essential. If $m \geq 2$, the strict upper interval is the proper nonempty Boolean poset on m elements, with classifying space S^{m-2}. It is k-connected exactly when $m - 2 > k$. Thus for finite k, the essential subsets are precisely those with $|S| \geq N - k - 2$; every subset is essential for unrestricted coefficients. Formula (3.10) gives (4.6), including leastness against all full subposets. The wedge calculation in lemma 4.2 independently displays the comma nerves of each proposed resolver.

Successive R_k add the nonempty rank $|S| = N - k - 3$, until R_{N-3} contains every object except a. The final step adds a. All inclusions in the stated chain are strict. □

The theorem does not rely on an unproved monotonicity principle for arbitrary full subcategories. Leastness is checked against every admissible competitor. It also avoids confusing a discrete-input calculation with a statement about all higher types: the proof uses every $A \in \mathcal{V}_k$ through lemma 3.5.

Corollary 4.4 (Unbounded length)**.** *For every $r \geq 1$, some finite indexing poset and fixed initial level have at least r distinct least coefficient-relative resolutions. Moreover,*

$$|\mathsf{A}_k(D)| = 1 + \sum_{j=1}^{k+2} \binom{N}{j} \quad (0 \leq k \leq N-3), \qquad |\mathsf{A}_{N-2}(D)| = 2^N. \tag{4.7}$$

Proof. Take $N \geq \max(3, r+1)$ and apply theorem 4.3. The count follows by complementation of the selected proper subsets. □

4.4 A compact family and the smallest first jump

The Boolean family records the ranks of explicit wedges, but exponential size is unnecessary for a long profile. Let Q_m be the ordinal sum of m two-element antichains, with layers $1, \ldots, m$ from bottom to top. A chain selects at most one vertex in each layer, so $\mathrm{B}Q_m$ is the join of m copies of S^0, hence S^{m-1}. These are the classical finite sphere models of McCord, studied in minimal-model theory by Barmak–Minian [16].

Proposition 4.5 (Linear-size profiles)**.** *For $m \geq 2$ put $C'_m = \{b < a\} \star Q_m$, with $D = \{b\}$. For $0 \leq k \leq m-2$, its least resolver consists of b and the last $k+2$ layers of Q_m, and has $2k+5$ objects. For $k \geq m-1$, including $k = \infty$, the least resolver is C'_m. Thus $2m+2$ objects realize m distinct least levels. For $m = 2$ this gives a six-object Set-to-groupoid jump, and no smaller finite poset with an initial one-object starting level can exhibit that jump.*

Proof. The strict upper interval of an element in layer $j < m$ is Q_{m-j}, with nerve S^{m-j-1}; the top layer has empty upper intervals. The interval above a has nerve S^{m-1}, while the interval above b has initial object a. The upper-interval criterion gives exactly the displayed levels. At every omitted object of a proper least resolver, the selected upper comma is Q_{k+2}, with nerve S^{k+1}.

For minimality of six, a strict Set-to-groupoid change requires some $x \neq b$ with connected but not simply connected strict upper nerve. A connected poset on at most three objects has contractible order complex: it is a point, a tree, or the filled triangle of a total order. The upper interval therefore has at least four objects. Adding the distinct objects x and b gives at least six. □

The compact calculation is an application of standard sphere models, not a claim to have discovered those models. It shows that the coefficient sensitivity is not an artifact of the size of a Boolean lattice. The larger family retains its separate quantitative use: the ranks of its wedges measure multiple simultaneous obstructions.

4.5 An explicit witness at the next coefficient level

Proposition 4.6 (Detection of missing coherence)**.** *For $0 \le k \le N-3$, the level R_k fails to $(k+1)$-resolve D. For $A = \mathrm{K}(\mathbb{Z}, k+1)$, the failure is visible at a: the target of (3.7) has connected components indexed by*

$$\mathbb{Z}^{\binom{N-1}{k+2}}, \tag{4.8}$$

whereas its source is connected. The constant-map comparison reaches only the zero component.

Proof. Put $q = k+1 \ge 1$. By lemma 4.2, the indexing space at a is $T \simeq \bigvee^h S^q$, where $h = \binom{N-1}{k+2} > 0$. Eilenberg–Mac Lane representability gives

$$\pi_0 \operatorname{Map}(T, \mathrm{K}(\mathbb{Z}, q)) \cong H^q(T; \mathbb{Z}) \cong \mathbb{Z}^h.$$

The space $\mathrm{K}(\mathbb{Z}, q)$ is connected. Its constant maps have zero degree on every sphere and belong to the zero cohomology class. Hence the comparison is not an equivalence. □

This is a cohomological witness to a homotopical failure, not an assertion that cohomology generally classifies homotopy types. The preceding wedge identification justifies the calculation. No inference from acyclicity to contractibility is used.

4.6 Relation to ordinary topos levels

At $k = 0$ we are in the ordinary presheaf topos $\mathrm{Fun}(\mathcal{C}_N, \mathbf{Set}) = \mathbf{Set}^{\mathcal{C}_N}$. Restriction to a full subposet has the triple (3.1), with left-exact middle functor, and induces an essential subtopos. Finite posets have no nontrivial retract identifications. Apply the finite-site rigidity result to $\mathcal{C}_N^{\mathrm{op}}$ [17]. Its subtoposes are induced by full subposets. More explicitly, the topology J_R declares a sieve at x covering exactly when it contains every arrow $r \to x$ with $r \in R$. The objects of R are exactly the J_R-irreducible objects. Thus inclusion of the selected subsets agrees with the subtopos ordering, and a level above D must include b. The set-valued member is therefore an ordinary Aufhebung calculation among all such levels. For higher coefficients the theorem retains its explicit full-subcategory qualification.

These posets are not a counterexample to a special cohesive-site theorem. In $\mathcal{C}_N^{\mathrm{op}}$, the terminal object is b, but there is no map $b \to x$ for $x \ne b$. The usual hypothesis that each site object has points therefore fails. The canonical components functor also fails to preserve binary products: representables at two distinct maximal faces each have singleton colimit, whereas their pointwise product is empty. Our comparison concerns the abstract resolution condition, not a refutation of a result with those additional cohesive hypotheses.

5 Examples: the loop and nonconstant generation

5.1 The eight-object Boolean member

Take $N = 3$. The six elements of $\mathcal{P}_3$ are the vertices and edges of a triangle, ordered by incidence. Their nerve is the subdivided boundary circle. The full category adds $b < a$

beneath them, as in figure 1. The theorem gives

$$\mathsf{A}_0(D) = \{b\} \cup \mathcal{P}_3, \qquad \mathsf{A}_1(D) = \mathsf{A}_\infty(D) = \mathcal{C}_3.$$

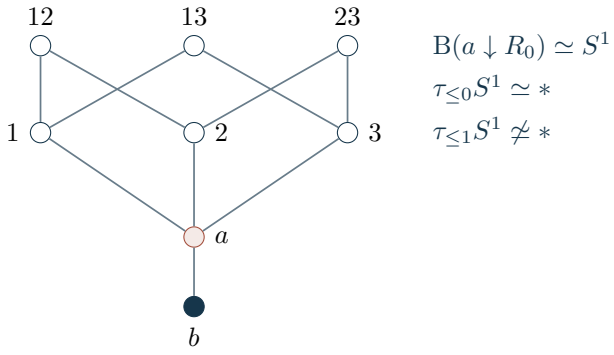


Figure 1: The eight-object poset $\mathcal{C}_3$. The set-valued least resolver contains every object except a; the space-valued least resolver also requires a. This is a Hasse diagram: arrows point upward and transitive arrows are omitted. The six incidence edges form a circle in the order complex, despite crossings in the drawing.

For a set A, compatible assignments of its elements over a connected indexing category are constant. Thus

$$\lim_{\mathcal{P}_3} \Delta A \cong A.$$

There is no other omitted-object test, since only a is omitted. Leastness also has a direct description. Every maximal edge object must be present: otherwise its comma category is empty. Each vertex must then be present: if it is absent, the two incident maximal edges give a disconnected comma category. All six objects are forced.

For a space A, however,

$$\lim_{\mathcal{P}_3} \Delta A \simeq \mathrm{Map}(S^1, A).$$

The comparison selects constant loops. Taking $A = \mathrm{K}(\mathbb{Z}, 1)$ produces components indexed by $\mathbb{Z}$. The earlier resolution did not require this loop data to vanish; it did not test it. Once groupoids become admissible, the loop becomes visible. Adding a gives the relevant comma category an initial object and repairs the failure.

Indexing size	Sets	Groupoids	2-types	3-types
$\lvert\mathcal{C}_3\rvert = 8$	7	8	8	8
$\lvert\mathcal{C}_4\rvert = 16$	11	15	16	16
$\lvert\mathcal{C}_5\rvert = 32$	16	26	31	32

Table 1: Cardinalities of the least resolving full subposets. The starting level and ambient indexing data are fixed within each row. Every row eventually stabilizes at the whole indexing category.

These numbers are not a measurement of philosophical progress. They count objects in a constructed poset. Their role is to distinguish senses of resolution that ordinary notation could otherwise conceal.

5.2 A nonconstant nine-object example

The constant generation of the main family makes its comparison unusually transparent. The same first jump survives genuinely nonconstant generation. Start with $p < a < \mathcal{P}_3$, and add an object q with $q < 1$, hence $q < 12, 13$, and no other new comparisons. Here $1, 2, 3$ denote singleton faces, while $12, 13, 23$ denote two-element faces. Let $D = \{p, q\}$; its two objects are incomparable. For a pair (A, B) of coefficient objects, left extension produces the diagram $X = \mathrm{Lan}_D(A, B)$ with

positions	value of X
$p, a, 2, 3, 23$	A
q	B
$1, 12, 13$	$A \sqcup B.$

(5.1)

The nonidentity transition maps are identities or the appropriate coproduct inclusions. The lower comma categories for generation are discrete, so these coproducts agree on old inputs in every coefficient category. The diagrams are not constant, even when A and B are connected nonempty types.

Proposition 5.1. *In this nine-object poset the least full resolver of D is $C \setminus \{a\}$ for sets, and C for all $k \geq 1$, including unrestricted spaces.*

Proof. Taking $B = \varnothing$ and A a two-element set forces all six faces: the matching map at an edge is $A \to *$, and at a vertex it is the diagonal $A \to A \times A$. Both fail to be bijective. Thus any resolver already contains $D \cup \mathcal{P}_3$.

To test the only remaining position a, restrict X to $J = \mathcal{P}_3$. Color its A summands by 0 and its B summands by 1, giving a natural map $X|_J \to \Delta_J\{0, 1\}$. Since $\mathrm{B}J \simeq S^1$ is connected, the limit of the latter is $\{0, 1\}$. The fiber of $\lim_J X$ over 0 is the limit of the constant A diagram, hence $\mathrm{Map}(S^1, A)$. Its fiber over 1 is empty: that fiber diagram has empty values at $2, 3, 23$. Limits preserve these pullbacks. Consequently

$$M_a X = \lim_J X \simeq \mathrm{Map}(S^1, A),$$

and its matching map is the constant-map comparison from A. For sets it is always an equivalence. For $k \geq 1$, take $A = \mathrm{K}(\mathbb{Z}, 1)$ and $B = *$: the comparison misses all nonzero winding components. Apply proposition 3.2 and corollary 3.3. □

This example isolates the two operations even without constant output from left generation: every earlier generated diagram is preserved, while the right reconstruction detects additional coherence.

6 What changes, and what remains true

6.1 Enlarging a test is not refuting its restricted version

There is an easy but misleading way to describe the example: homotopy theory reveals that the set-valued resolution was false. It does not. The set-valued assertion remains true for every set. What changes is the range of objects over which sufficiency is quantified.

Proposition 6.1 (Persistence of the earlier tests)**.** *In the setting of lemma 3.5, let $0 \leq k \leq \ell \leq \infty$. For every $A \in \mathcal{V}_k$, the resolution comparison is the same when computed in $\mathcal{V}_k$ or in $\mathcal{V}_\ell$. Hence an ℓ-resolver is a k-resolver, while a k-resolver can fail to be an ℓ-resolver only on additional test objects.*

Proof. The inclusion $\mathcal{V}_k \hookrightarrow \mathcal{V}_\ell$ is fully faithful and preserves limits. Left extension from D is constant in both settings. Thus the pointwise right Kan extension and its unit agree on $A \in \mathcal{V}_k$. An equivalence for all objects of the larger category is, in particular, one for all objects of the smaller category. □

This proposition locates the result exactly. It does not exhibit a contradiction between two answers to the same fully specified mathematical question. It exhibits different answers to related questions whose coefficient parameter was easily left implicit. The philosophical problem is to determine when that parameter may legitimately remain fixed and when the interpretation itself requires it to vary.

In particular, our main family is stronger as a test than the familiar observation that a homotopy colimit can differ from an ordinary colimit. That observation is relevant background [18], and an elementary example is recorded in the appendix. Here the source construction is unchanged even on nondiscrete coefficients: $\mathrm{Lan}_D A$ is constant. The defect lies in its reconstruction through a higher right Kan extension. We have isolated the test of resolution from a possible change in the generation of the object being tested.

More generally, let $i : \mathcal{V} \hookrightarrow \mathcal{W}$ be fully faithful and preserve the relevant limits. If the canonical comparison

$$i\,\mathrm{Lan}_D^{\mathcal{V}} A \longrightarrow \mathrm{Lan}_D^{\mathcal{W}}(iA)$$

is an equivalence for every old input A, then the generated resolution units agree on those inputs. For finite C, this implies $\mathsf{A}_{\mathcal{V}}(D) \subseteq \mathsf{A}_{\mathcal{W}}(D)$: a resolver in $\mathcal{W}$ passes every old comparison, and hence contains the least one in $\mathcal{V}$. The initial-singleton case satisfies this hypothesis, as does the nine-object example because its generation uses coproducts. The hypothesis is sufficient, and the next result shows why it cannot simply be omitted.

6.2 Changing generation can shrink a least resolver

Preservation of left generation is a substantive hypothesis in the comparison of coefficients. Without it, the least resolving subset can become smaller when the coefficient category becomes larger.

Proposition 6.2 (Reversal when generation changes)**.** *Let J be a nonempty finite poset, and put*

$$C = \{x\} \star J \star \{t\}, \qquad D = J, \qquad U = J \star \{t\}.$$

For finite $k \geq 0$ let $T_k = \tau_{\leq k}\mathrm{B}J$, and put $T_\infty = \mathrm{B}J$. Then

$$\mathsf{A}_k(D) = \begin{cases} C, & T_k \simeq *, \\ U, & T_k \not\simeq *. \end{cases}$$

In particular, if $J = Q_{d+1}$ is the ordinal sum of $d+1$ two-element antichains, where $d \geq 1$, then $|C| = 2d + 4$ and

$$\mathsf{A}_k(D) = \begin{cases} C, & 0 \leq k < d, \\ C \setminus \{x\}, & d \leq k \leq \infty. \end{cases}$$

Proof. For every $A : J \to \mathcal{V}_k$, its left extension $X = \mathrm{Lan}_D A$ has $X(x) = \varnothing$, restricts to A on J, and has $X(t) = \mathrm{colim}_J A$ with the canonical cocone. The terminal position t is mandatory at every coefficient level: taking the empty input diagram gives the matching map $\varnothing \to *$ at t. The positions in D are mandatory by definition. It therefore remains only to decide whether x is mandatory.

Take the constant terminal input, and write $F = (\mathrm{Lan}_D 1)|_U$. Its restriction to J is terminal and $F(t) = T_k$. The canonical cocone represents the truncation map $\eta : \mathrm{B}J \to T_k$, or the identity for $k = \infty$. A cone over F is a cone over its restriction to J, a point of T_k, and compatibility with that cocone. Consequently

$$\lim_U F \simeq T_k \times_{\mathrm{Map}(\mathrm{B}J, T_k)} \{\eta\},$$

where $T_k \to \mathrm{Map}(\mathrm{B}J, T_k)$ sends a point to the corresponding constant map. By the universal property of truncation, $\mathrm{Map}(\mathrm{B}J, T_k) \simeq \mathrm{Map}(T_k, T_k)$; under this equivalence, η is the identity and constant maps remain constant. The displayed homotopy pullback is therefore nonempty exactly when the identity of T_k is homotopic to a constant, equivalently when T_k is contractible.

If T_k is noncontractible, this limit is empty. For arbitrary A, the unique map $A \to 1$ induces

$$\lim_U (\mathrm{Lan}_D A)|_U \longrightarrow \lim_U F = \varnothing.$$

An object of spaces or truncated spaces admitting a map to the empty object is itself empty. Thus the matching map at x is $\varnothing \to \varnothing$ for every input, and U resolves D. Together with the mandatory positions already identified, this proves leastness.

If T_k is contractible, F is pointwise terminal on U, so its limit is terminal. The matching map at x for the terminal input is then $\varnothing \to *$. Thus x is also mandatory, and the least resolver is C.

Finally, $\mathrm{B}Q_{d+1} \simeq S^d$. Its k-truncation is contractible exactly for $0 \le k < d$, and is noncontractible for finite $k \ge d$ and for unrestricted spaces. This proves the last formula. □

For $d = 1$, the least resolver has six objects for sets and five for groupoids and all higher coefficients. This reverses the direction of the main family's change. The reason is explicit: the terminal input is generated with value $*$ at t in sets and S^1 at t in groupoids. The main family and the nonconstant nine-object example preserve their left generation on old inputs; this example does not. The two behaviors concern different hypotheses.

Six-object family	$\lvert D\rvert$	Sets	Groupoids	Generation on old inputs
C_2' of proposition 4.5	1	5	6	Preserved
Cone family of proposition 6.2	4	6	5	Changed

Table 2: Sizes of least resolvers in two different six-object posets. Preserving generation guarantees monotonicity under the stated comparison hypotheses; changing generation can reverse the inclusion.

6.3 A resolution profile

For a fixed ambient indexing category, starting level, and admissible family, call the sequence $k \mapsto \mathsf{A}_k(D)$ its *resolution profile*, where the least levels exist. The expression is a name for the comparison being studied, not a replacement for established cofinality terminology.

The profile contains more information than a single result. For the growing families it shows both the validity of a restricted resolution and the first expressive level at which it ceases to suffice. The reversal profile instead loses a formerly essential position when generation changes. In theorem 4.3, the failure has a specified location, namely a, a specified

shape, a wedge of spheres, and a specified detecting coefficient, an Eilenberg–Mac Lane space. What would otherwise be an unspecified demand for "more structure" becomes a calculable request.

This also clarifies a distinction between erasure and invisibility. The nerve is a circle already in the first example. Set-valued coefficients do not turn that nerve into a contractible space; they are unable to distinguish it from a point by the maps under consideration. The distinction untested by the comparison is therefore relative to that test. To say that a result preserves everything relevant is to say something about what counts as a relevant test, not only about the construction of the result.

6.4 Why leastness is not enough

A least object is uniquely determined up to the relevant equivalence once the ordered collection of candidates and its predicate of adequacy are fixed. Leastness can therefore express a strong form of internal necessity. In our family no alternative full subposet is smaller and still meets the specified test. The matching-support calculation excludes all such alternatives.

Consider an explicit inference, introduced here as a claim to be tested, not attributed to any of the authors discussed: if an enrichment keeps the generation of the diagrams and every earlier resolution comparison intact, then it also keeps their least adequate resolving level intact. The premise is stronger than agreement on selected examples. It preserves the entire old domain of comparisons. Yet proposition 6.1 and theorem 4.3 jointly refute the conclusion, with arbitrarily long finite chains of different least levels.

What survives is correctness on the earlier domain; what fails is its sufficiency to determine the least adequate reconstruction on the enlarged domain. This distinction is not a discrepancy between an informal reading and a formal calculation. It occurs between formally stated properties, with fixed indexing data and fixed generation. Leastness combines an order of candidates with a predicate of adequacy. Persistence of old comparisons preserves one part of the predicate's evidence without establishing the completeness of its range.

For a reconstruction of dialectical necessity, the consequence is conditional but exact. If its explanatory commitments give a reason to admit additional tests, continued agreement on all earlier tests does not by itself secure its earlier leastness claim. It must justify the original test domain, prove a transport result, or accept a changed least resolution. If no such reason is given, our theorem supplies a possible comparison rather than a refutation. This is the distinction between a useful external analysis and an internally demanded revision to which we now turn.

7 Determinate preservation as an interpretive requirement

7.1 A criterion with a limited demand

We propose the following requirement for a mathematical reconstruction of sublation:

> State the relations claimed to represent the mediation and the distinctions whose retention constitutes the proposed preservation; show how the resolving construction retains them; and justify the treatment of distinctions that become detectable under the reconstruction's own licensed changes of language.

This requirement of *determinate preservation* is our methodological proposal. It is not a

quotation from Hegel or a theorem of category theory. Its motivation is that preservation needs a determinate content if it is to play an explanatory role. The theorem supplies an exact case in which an unqualified preservation claim would conceal a choice.

The demand is intentionally weaker than invariance under every enrichment. Not every new formal distinction deserves philosophical recognition. A set-based model can be exactly appropriate for a question about ordinary classification. A groupoid model may be appropriate when identifications and automorphisms matter. The requirement is to defend this choice in relation to the interpreted material, not to rank models by how much machinery they contain.

A change of test becomes relevant in at least three distinguishable ways. The reconstruction may itself use the additional structure to explain the passage; its claimed applications may require that structure; or a close reading may show that its original tests identify distinctions it needs to retain. These are warrants that can be contested with textual and mathematical evidence. Merely knowing that groupoids are available is not such a warrant. Conversely, declaring that only sets are admissible does not settle the issue if the explanatory argument already invokes coherent identity. The present theorem diagnoses consequences once a warrant is given; it does not manufacture that warrant.

Nor does the requirement insist that every revision add more objects. In the generation-preserving families the least resolver grows; proposition 6.2 shows that a change of generation can instead make it shrink. Another model might revise its equivalences, its morphisms, or its admissible tests instead. The general lesson is about explicit dependence on representational choices; the particular repairs in a constructed poset do not constitute a universal recipe for dialectic.

7.2 Retention without identity of status

The philosophical attraction of sublation is that something can persist without retaining its former independence. A categorical reconstruction can give this thought more than one precise form. An object may occur as the image of an embedding, as a retract, as a component of gluing data, or as the value of a modality. These are different modes of dependence. Their differences should be used rather than obscured by a common label.

Our resolution condition does one specific job: an earlier skeletal object is recoverable through a later right extension. This captures a relation of recoverability and changed presentation. It does not independently capture every philosophical sense of negation, nor does it show that the earlier concept has criticized itself. The model is informative because its achievement can be stated exactly.

There is also a limit to the familiar picture of preservation as storage. Keeping a copy of every earlier object need not preserve the relations that made those objects what they were. In homotopy theory, retaining points while suppressing paths can alter the character of the object. In an inferential account, retaining sentences while changing which consequences they license can alter their content. The analogy between these cases is methodological, not an identification of paths with inferences. Both require the interpretation to specify the relations its preservation concerns.

7.3 What is established in the content, and what is available to us

The distinction in Hegel's discussion of determinate being is especially instructive. In online paragraph 193, a determination may be available to our reflection before it is established as part of the concept's own development. Such a comparison can elucidate or anticipate;

that does not give it the status of a ground of the transition [12].

The author's running commentary, distinct from the earlier interactive map, foregrounds this problem in its first two readings [22]. Its distinction between what is available "for us" and what has been posited within the development helps formulate the interpretive demand. It is a statement of the problem, not independent evidence that a particular reconstruction meets it.

Applied to this paper, the distinction asks why a coefficient enrichment belongs to the represented content rather than only to our means of investigating it. The answer cannot be that the richer formalism exists. Nor can it be that the enrichment has produced an interesting theorem. Those facts justify studying the comparison, but they do not determine its interpretive status. A close reading must identify the relevant demand of the content; a mathematical preservation theorem must then establish what the chosen formal correlate achieves.

This yields an interpretation audit with a definite result. Measured against the demands drawn from the becoming passage, our formal calculation establishes recoverability and least repair only for its stipulated mathematical data. It does not supply the moments' self-transformation or explain their altered significance. For the status of the coefficient category, it supplies a sensitivity result but no immanent warrant for enrichment. The mathematical result is therefore philosophically useful in an elucidatory role, while its promotion to a derivation would require evidence it does not provide.

7.4 Immanence and the status of the formal language

Hegel's final discussion of method presents the relation between method and content as internal rather than as the application of an independently chosen technique [19]. A mathematical reconstruction necessarily begins with some formal resources. It cannot escape that fact by describing those resources as universal. It can, however, make their role an object of the reconstruction rather than leaving it as an unexplained background.

Hentrup's account of rational proof supplies an independent philosophical comparator. In section II, geometric demonstration can establish a necessity belonging to its object while its progression remains the prover's activity. In sections V–VI, rational proof requires mutual mediation in which the terms' identities develop, rather than reciprocal support between fixed terms [20]. On this reading, the difference between valid demonstration and adequate mediation is already a substantive Hegelian problem. Our contribution is not to discover that difference, but to make two dependencies of a formal reconstruction calculable: the tests that determine its least resolver and the relations whose interpretation remains to be justified.

The resolution profile offers one finite way to do so. The coefficient category is first a parameter of the model; the profile then shows how the model's verdict depends on it. This does not reproduce Hegel's claim that the method determines itself. It provides a disciplined partial analogue: the consequences of the choice of method are brought into the argument and can be evaluated. A stronger claim of self-determination would require additional work explaining why the sequence of admissible tests is generated by the subject matter rather than chosen by us.

The gap should remain visible. Our theorem can force the next resolving level *if* the next coefficient category is admitted. It does not force the admission of that category. Confusing these two steps would repeat, at a higher level, the very substitution of external comparison for internal development that the reconstruction is meant to illuminate.

7.5 A reader audit: stable profiles and different relations

Coefficient variation is not the only choice a reconstruction makes. The author's earlier interactive reader [21] supplies a concrete case in which to distinguish it from the choice of indexing relations. The reader locates sections within larger sections and opens their associated passages. Containment and succession are both legitimate editorial relations. Neither, by itself, establishes the dependence by which one determination generates another.

Consider two explicitly stipulated encodings of the same positions. In one, an ancestor is below its descendants in a rooted tree. In the other, the positions are totally ordered by a traversal compatible with that ancestry. A synthetic book-header root is first in both. It is an editorial device, not an identification with Pure Being; likewise the second order records succession, not implication.

Proposition 7.1 (A limit of the profile diagnostic)**.** *Let T be a finite rooted tree with root b, with at least two children at every nonleaf vertex. Let L be a total order on the same vertex set extending the ancestor order, and write t for its last vertex. For $D = \{b\}$ and every $0 \le k \le \infty$,*

$$\mathsf{A}_k^T(D) = T, \qquad \mathsf{A}_k^L(D) = \{b, t\}. \tag{7.1}$$

Thus both profiles are invariant under coefficient enrichment, although their least subsets can differ by arbitrarily many objects.

Proof. At a leaf of T the strict upper interval is empty. At a nonleaf it is the disjoint union of the descendant subtrees rooted at its children. Each component has an initial object, so its nerve is contractible; there are at least two such components. Every vertex therefore belongs to $E_k(T)$ at every k. In L, only t has empty strict upper interval; at every earlier vertex that interval has a least element. Hence $E_k(L) = \{t\}$. Apply theorem 3.6 and add the required starting object b. Finite branching trees can have arbitrarily many vertices. □

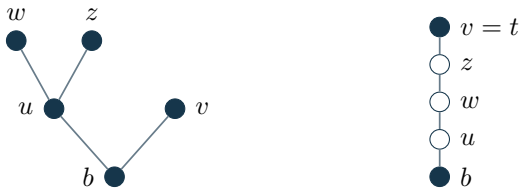


Figure 2: Two stipulated orders on the same positions. Filled vertices are in the least resolver for every coefficient level. The invariance of both profiles does not decide which relation, if either, models the intended mediation. Arrows in these Hasse diagrams point upward.

The active reader contains 120 positions in four displayed depths. After excluding its one unnamed placeholder and adding the synthetic root, there are 120 named-or-root positions. Every internal vertex still has at least two children. The proposition therefore applies to these proposed encodings. These counts describe the inspected editorial data; they assert neither a canonical inventory of Hegelian categories nor a mathematical structure derived from the text.

The contrast supplies a limit on our own diagnostic. Invariance of a resolution profile is insufficient for interpretive adequacy. The opposed leastness verdicts here are explained by

branching and terminality before any philosophical content is assigned to the labels. This does not make either encoding defective: it specifies what each achieves. In this case the unsettled choice concerns relations; in the Boolean family the relations are fixed and the varying choice concerns tests. The two choices require distinct arguments.

A passage-grounded extension would therefore distinguish containment, succession, and argued dependence, recording textual evidence for the latter instead of silently replacing it with either of the former. Composites and identifications between dependency paths would require further justification before a category or higher category is claimed. The present paper does not invent those relations in order to find Hegelian loops. It identifies what such an extension would need to establish before a homotopy calculation acquired that interpretation.

7.6 A mathematical reading need not be a reduction

The distinction between orientation and derivation has a direct textual basis. In the final discussion of method, Hegel permits an advance survey of the whole while denying that division into its already specified parts establishes their transition (online paragraph 1813). The author's earlier interactive reader [21] makes this distinction concrete: its hierarchy locates moments and opens the corresponding passages. Such navigational success need not claim to establish their mediation. Nor does a circular display by itself realize the return through mediation described in paragraph 1814. A formal interpretation must say which of these achievements it intends.

There is also a challenge to our own procedure. Paragraph 1812 rejects the claim that criticism of cognition's instrument enjoys an automatic priority over engagement with the subject matter: that criticism has presuppositions of its own [19]. Determinate preservation should therefore operate within the examination of a reconstruction, rather than as a preliminary tribunal that settles its possibility in advance. The test must answer to the content it claims to illuminate. Our theorem identifies a dependence to be justified; it neither supplies a presuppositionless standpoint nor decides that justification before the interpretation has been attempted.

The value of a mathematical reading is not exhausted by translation into a fixed formal calculus. It can reveal a question that was difficult to state, separate claims previously run together, and supply counterexamples to an overextended argument. Here the precise separation is between the existence of a least resolver and the justification of its criterion of resolution.

This supports a reciprocal rather than a reductive relation between philosophy and mathematics. The philosophical concern with retention motivates a comparison of coefficients; the mathematical comparison then forces the philosophical concern to become more specific. Neither side receives authority simply by borrowing the other's vocabulary. What matters is whether the encounter changes the questions that can responsibly be asked and the answers that can be defended.

8 Contemporary alternatives and adjacent results

8.1 Progression, contradiction, and logical form

Priest's model uses paraconsistent logic, but also specifies a concrete retention condition: introducing the new predicate leaves the old vocabulary's extensions and anti-extensions unchanged [23, section 3]. In section 6 he expressly limits the aim to the logical structure of

progression, rather than an explanation of why a progression occurs. These are precedents for specifying preservation and scope, not errors our theorem corrects. His retention condition is stipulated in the construction; ours is Kan-extension recoverability with a proved leastness profile. Neither identification alone establishes its adequacy to the text.

An adjunction does not assert $P \wedge \neg P$, and a homotopical obstruction does not license or reject explosion. A useful comparison must therefore identify shared explanatory tasks before treating paraconsistency and modal adjunctions as competing answers to one question.

Yang's published abstract distinguishes the accuracy of describing dialectical progression from the ability to explain it [24]. This distinction is congenial to our separation of retention, determination, and necessity. We do not rely on a detailed reconstruction of Yang's argument, whose full text was not available for this study. Our mathematical result provides an independent instance of the general difference: a diagram can describe a proposed transition without settling why its adequacy criterion is the appropriate one.

Kolman's account treats Hegel's relation to logical form through the reconsideration of fixed determinations rather than the mere addition of one more logical form [25]. Our proposal shares that concern with the status of a formal standpoint, while giving it a different technical expression. Changing coefficients does not model a history of logic. It gives a controlled comparison in which what a formalism can detect becomes part of the account of its verdict.

Brown offers a positive example of making preservation explicit. Definition 2.6 calls a morphism $f : X \to Y$ of incompatibility frames conservative when every finite multiset Γ is incompatible in X exactly when $f(\Gamma)$ is incompatible in Y. Proposition 2.1 proves this property for the adjunction unit; the signed construction inherits it in section 3.2 [26]. The universal construction thus has a separately stated preservation achievement. It differs from our Kan-extension recoverability. The methodological point is that specifying an adjunction and specifying what it retains are distinguishable tasks.

8.2 What recent Aufhebung calculations already achieve

The work of Kennett, Riehl, Roy, and Zaks computes actual skeletal and coskeletal bounds in ordinary presheaf topoi [2]. Their results show why an unspecified "next dimension" is not an adequate description of Aufhebung. The precise bound depends on the indexing category, and low-dimensional cases require attention.

The recent solution by Hora, Kamio, and Maehara to Lawvere's fourth problem computes the relation for symmetric simplicial sets [27]. This removes that calculation from the list of open targets. Our family changes a different parameter: its indexing category is fixed while its coefficients vary. The elementary topology involved does not make it a replacement for the substantially deeper combinatorics of their theorem.

Menni's work on the least subtopos containing the discrete skeleton of Ω shows the power of replacing a large containment condition by a distinguished generator [3]. It also makes essentiality a substantive issue. One cannot freely identify a least subtopos with a least essential resolving level. This is why our higher theorem names its admissible full-subcategory levels explicitly instead of relying on an unstated lattice classification.

Dey–Lesnick and Brüstle–Desrochers–Leblanc address the least full subposets needed for ordinary limit or colimit computation [5, 14]. They are direct antecedents of the $k = 0$ upper-interval mechanism, while their treatment of nonfull relation-sparse subposets optimizes a different admissible class. Our calculation combines a truncated version of this mechanism with explicit sphere obstructions to compare coefficients. It does not reopen the ordinary

least-full-subposet problem as if that literature did not exist.

8.3 What a checked construction preserves

DeBruce's Cubical Agda tutorial offers a useful contemporary example because it places explicit limitations beside its formal declarations. Chapter 4 introduces a higher inductive type `Becoming` with two points, `being` and `nothing`, and a path between them [28]. These points are new inhabitants of a common carrier. They are not the unit and empty types used for Being and Nothing in Chapter 2. The distinction matters: a path between those earlier types would transport an inhabitant of the unit type into the empty type. The consistent construction therefore depends on changing what is related, not simply on replacing judgmental equality with path equality. An interpretation must explain which roles of the earlier assignments survive this change.

The same chapter declares an `Aufhebung` record containing a higher unity and a function from the lower moment to the higher. It explicitly leaves an instance unconstructed and acknowledges missing compatibility requirements; it supplies no leastness theorem. These are useful limits on the achievement, not reasons to dismiss it. The tutorial makes an interpretive audit concrete: naming, providing a structure, retaining a specified property, and proving minimality are distinguishable accomplishments. We assess the inspected declarations and their interpretation; we have not independently rerun that project's Agda build.

8.4 Higher modal semantics is infrastructure, not a transfer theorem

Uemura's internal languages for diagrams of infinity-logoses connect modalities, gluing, and higher categorical semantics [29]. Shulman's multimodal adjoint type theory provides a systematic treatment of modal adjunctions and their semantics [30]. Myers and Riley study multiple commuting cohesions [31]. These works make the higher setting substantially more precise than an informal instruction to replace sets by types.

The computational side is also important. Gratzer's normalization result relates type checking to equality in the chosen mode theory [32]; the combination of cubical and multimodal type theory provides another route toward expressing homotopical modal arguments [33]. None of these metatheorems selects the mode theory appropriate to Hegel, and none by itself says that a given set-valued resolution has the same least solution in spaces. They supply languages in which such questions can be asked rigorously.

Protin's recent discussion of Hegel and topology offers mathematical illustrations and hermeneutic proposals [34]. The contrast with our purpose is one of task: we isolate a preservation claim and calculate its failure of transport. An illustration can illuminate a passage without claiming to establish a universal property; a universal property can be proved without establishing that the passage has been interpreted correctly. The distinction protects both kinds of work from exaggerated expectations.

9 Conclusion: the conditions of a necessary transition

The finite families give definite mathematical answers. A fixed initial level can have successively different least resolutions when its admissible coefficients range from sets through higher truncated spaces. The entire sequence is calculable; its obstructions are explicit; its leastness holds against every full-subcategory competitor. The earlier answer remains true within its domain, while a richer domain detects what it did not test. When generation

changes as well, the reversal family shows that the least resolver can instead shrink. Preserving the earlier generation is therefore a substantive condition on a comparison, not a harmless convention.

The reader audit supplies the converse restraint. Containment and succession, encoded on the same positions, can yield sharply different least resolvers while both profiles remain coefficient-invariant. Robustness under a change of tests is therefore no certificate that the relations being tested express the intended mediation. Together the examples separate two tasks: accounting for the admitted relations, and accounting for the domain over which their adequacy is tested.

For a mathematical reading of Hegel, the result recommends neither a retreat from formalization nor an unqualified preference for higher structures. It recommends a more demanding account of preservation. The interpretation must identify the content of what survives, the equivalences under which it survives, and the tests by which its survival is judged. As the interpretation proceeds, those choices must answer to the text and explanatory task. A universal property can then help explain the conditional necessity of its result without being asked to justify the representation by itself.

Lawvere's approach is especially valuable because it permits this demand to become mathematics. A proposed resolution can fail, and the failure can have a shape. In our example that shape is a sphere invisible to one coefficient category and visible to the next. The philosophical lesson is not that every hidden sphere is a dialectical moment. It is that a formal account of a necessary transition owes an account of what its notion of necessity can see.

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A The preliminary left-extension obstruction

An elementary calculation motivated the main construction but should be distinguished from it. Let $J = \mathcal{P}_3$, and add a terminal object t to obtain C. For the constant singleton diagram $1 : J \to \mathbf{Set}$,

$$(\mathrm{Lan}_J^{\mathbf{Set}} 1)(t) = \operatorname*{colim}_J 1 = 1.$$

For the same discrete-valued diagram in spaces,

$$(\mathrm{Lan}_J^{\mathcal{S}} 1)(t) \simeq \operatorname*{hocolim}_J * \simeq \mathrm{B}J \simeq S^1.$$

Thus the natural comparison from the space-valued extension to the discrete image of the set-valued extension is not an equivalence. This is a standard example of the topology retained by a homotopy colimit [18]. It is not, on its own, a failure of an Aufhebung relation. The main family isolates the latter kind of issue by making its left extension constant in every coefficient category.

More generally, for a set-valued diagram F on an ordinary category J, the homotopy colimit of its discrete image is the classifying space of its category of elements. The ordinary colimit records the connected components of that category. The comparison is an equivalence precisely when each component has contractible classifying space. This familiar description is useful for checking where a discrete-to-homotopy comparison can fail; it must not be promoted into a general contractibility test based on homology alone.

B Reproducibility and the status of verification

The accompanying program `verify_family.py` uses only the Python standard library. It constructs order complexes, represents boundary matrices as bitsets, and computes their ranks over $\mathbb{F}_2$ by Gaussian elimination. The report is saved as `verification.json`. Run it with:

```
python verify_family.py
```

For $N = 3$, the program examines all $2^7 = 128$ full subposets containing b. For $N = 4$, it examines all $2^{15} = 32{,}768$. At each omitted object it tests the necessary condition of connectedness and vanishing reduced homology through degree k in the comma nerve. The resulting candidate counts are:

N	$k = 0$	$k = 1$	$k = 2$
3	2	1	1[a]
4	32	2	1

[a]The script exhausts $k = 0, 1$ for $N = 3$; the last value follows from the identity-only result.

In each enumerated case, the homological necessary conditions agree with containment of the predicted least level. This agreement is supplementary evidence. It does not generally turn a necessary homological test into a sufficient homotopical one. The proof of sufficiency is the wedge calculation in the text.

For $N = 3, 4, 5, 6$, the program additionally checks the predicted homology of every proposed proper least resolver at a, verifies the expected failure at the next homological degree, and tests deletion of each mandatory face even when all other objects are available. It does not enumerate every subposet for $N = 5$ or 6.

The additional program `verify_matching.py` independently enumerates finite limits as compatible tuples. It examines every naturally labelled poset on up to four objects and every functor whose object sets have cardinalities $0, 1, 2$, checking all full reconstruction subsets. It performs 121,774 comparisons across 7,766 diagrams; the four-object cases include 5,763 diagrams with varying cardinalities. These are direct Set-valued checks of proposition 3.2, not nerve or homology tests. The program `verify_nonconstant.py` checks all 128 admissible subsets of the nine-object example against nine input pairs of finite sets, performing 1,152 reconstruction checks.

For the shrinking family, `verify_reversal.py` enumerates 553 Set-valued diagrams on Q_2, forms their quotient-set colimits, and performs 2,212 direct reconstruction comparisons. Only the whole six-object category resolves all these inputs. The higher claim rests on the written canonical-cocone and map-to-empty argument, not on this Set-valued enumeration.

The program `verify_models.py` checks the compact sphere models through eight layers and separately enumerates all naturally labelled initial posets on up to six objects. For the latter check, each relevant upper interval has at most four vertices. Its order complex is explicitly collapsed to a graph, whose cycle rank decides simple connectivity. No jump occurs below six objects; among the 357 naturally labelled six-object cases, exactly one exhibits the first jump. This does not replace the lower-bound proof in the text.

Finally, `verify_reader.py` audits the supplied `reader-positions.json`. It confirms the branching hypothesis for the tree comparison: 120 positions including the synthetic root, 80 leaves, one vertex with two children, and 39 with three. The data record editorial containment and an explicitly chosen traversal, not textually established relations of logical dependence.

All programs use only the standard library. Their reports carry the corresponding prefix and `-verification.json` suffix, except the original `verification.json`. Running each program from the source directory regenerates its report.

The mathematical argument in this manuscript is a written proof, supported by these finite calculations. It has not been checked in a proof assistant. The counts and homology computations should not be described as a machine verification of every homotopy-theoretic assertion.

AI tools assisted with the literature study, drafting, mathematical derivation, and verification code. Two Astra review agents independently criticized the mathematical and philosophical arguments, prompting substantive revisions. These checks constitute AI-assisted internal review, not external human peer review or proof-assistant certification.

C Questions opened by the calculation

For nontrivial starting levels, corollary 3.3 settles existence of a least full resolver over a finite poset. Its formula can still be difficult to evaluate. Left extension need not be constant, and its interaction with right extension introduces more than the homotopy type of an unweighted comma nerve. A further problem is to identify finite detecting families for the matching-map tests in (3.5), and conditions under which ordinary resolution transports to higher coefficients.

There are no incomparable minimal full resolvers for any starting level in the finite-poset setting of corollary 3.3. To seek such examples one must change the indexing or admissible class, for example by admitting nonidentity endomorphisms or restricting which full subcategories count as levels. The ordinary constant-diagram case also belongs to existing minimal-subposet theory [5, 14]. Neither formula supplies a general effective contractibility algorithm: homology alone does not decide the unrestricted condition.

A third question concerns the admissible level family. The ordinary finite-site case has a rigidity theorem, but no such classification is assumed for the higher coefficient categories here. Establishing a correspondence between the full-subcategory profiles and a broader notion of higher essential levels would strengthen the result. It should be proved rather than absorbed into terminology.

Finally, the textual benchmark in this paper can be developed into a full comparison of alternative models of the becoming transition. That would require accounts of the moments' self-transformation and altered significance beyond the recoverability test used here. The present calculation supplies a precise constraint on such an interpretation; it does not supply those missing accounts.